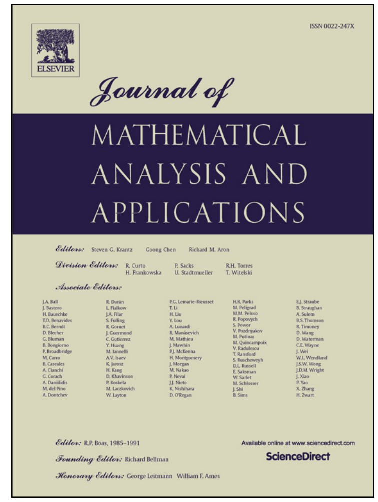
Accepted Manuscript



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PII: S0022-247X(18)31019-9
DOI: https://doi.org/10.1016/j.jmaa.2018.11.073
Reference: YJMAA 22759

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 5 February 2018

A fixed point result in Banach algebras based on the degree of nondensifiability and applications to quadratic integral equations

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Abstract

We present some fixed point results in Banach algebras based on the so called degree of nondensifiability ϕ_d. It is shown that ϕ_d is an alternative method to measures of noncompactnes to obtain fixed point result. As an application of the usefulness of ϕ_d it is proved the existence of solution for some quadratic integral equations.

Keywords: Banach algebras, Fixed point theorem, Degree of nondensifiabilty, Quadratic integral equations
2010 MSC: Primary: 47B48, 47H10, Secondary: 47H09, 45G10

1. Introduction

Quadratic integral equations are used to model many problems arising in diverse fields of applied science and engineering, as we can see for instance in [1, 2, 3] and references therein. In an abstract form, such equations can be written as

$$x = A(x)B(x) + C(x), \quad x \in \Omega, \tag{1.1}$$

where $A(x)$, $B(x)$, $C(x)$ are continuous mappings (below it will be specified) on a closed convex set Ω of a Banach algebra. To solve (1.1) it is usually needed to apply fixed-point results in Banach spaces. These results are based on the so-called *measure of noncompactness* (briefly MNC), see ([4, 5, 6, 7, 8, 9, 10, 11, 12]).

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To set the notation, $(X, \|\cdot\|)$ will denote a Banach space, B_X the closed unit ball and $\mathfrak{B}_X$ the class of non-empty and bounded sets of X. Likewise, given $S \subset X$, $\bar{S}$ and $\mathrm{Conv}(S)$ will represent the closure and the convex hull of S, respectively. As usually, $I := [0,1]$ and $\mathbb{R}_+ := [0, +\infty)$.

Since the definition of MNC may slightly vary according to the author (see, for instance, [13, 14]), here we adopt that of it was given in [15]:

Definition 1.1. *A mapping $\mu : \mathfrak{B}_X \longrightarrow \mathbb{R}_+$ is said to be a MNC if it satisfies the following properties:*

(i) *Regularity: $\mu(S) = 0$ if, and only if, S is a precompact set.*
(ii) *Invariant under closure: $\mu(S) = \mu(\overline{S})$, for all $S \in \mathfrak{B}_X$.*
(iii) *Monotony: $\mu(S_1 \cup S_2) = \max\{\mu(S_1), \mu(S_2)\}$, for all S_1, $S_2 \in \mathfrak{B}_X$.*
(iv) *Semi-homogeneity: $\mu(\lambda S) = |\lambda| \mu(S)$, for all $\lambda \in \mathbb{R}$ and $S \in \mathfrak{B}_X$.*
(v) *Invariant under translations: $\mu(x+S) = \mu(S)$, for all $x \in X$ and $S \in \mathfrak{B}_X$.*

A widely studied MNC is that of Hausdorff, denoted by χ (see, for instance, [13, 14]) and defined as

$$\chi(S) := \inf\left\{\varepsilon > 0 : S \text{ can be covered by finitely many balls with radii } \leq \varepsilon\right\},$$

for every $S \in \mathfrak{B}_X$. For instance, if X has infinite dimension, then $\chi(B_X) = 1$. Moreover, χ is invariant under the passage to the convex hull, that is, $\chi(\mathrm{Conv}(B)) = \chi(B)$ for any $B \in \mathfrak{B}_X$.

Remark 1.1. *In others papers (for instance [4]), monotony condition is given by*

$$(iii)' \quad \mu(S_1) \leq \mu(S_2), \quad \textit{for all } S_1, S_2 \in \mathfrak{B}_X \textit{ with } S_1 \subset S_2.$$

However, $(iii)'$ has been substituted by (iii) which is more restrictive than $(iii)'$ since $(iii) \Rightarrow (iii)'$. It is immediate that regularity condition, (i) in Definition 1.1, does not hold for MNC satisfying $(iii)'$ instead of (iii). Indeed, $\mu(S) :=$ $\mathrm{Diam}(S)$ proves this fact since precompactness of S does not implies necessarily $\mu(S) = 0$. Note that Hausdorff MNC χ satisfies condition (iii) so $(iii)'$.

In many works (see, for instance, [8, 11]), a key fact in the proof of fixed point results is, under suitable conditions, to start proving the existence and continuity of a mapping $T := (\frac{\mathcal{I}-C}{A})^{-1}$ defined on $B(\Omega)$, $\mathcal{I}$ being the identity, A, C given in (1.1), and applying then to T the celebrated Darbo fixed point theorem (see, for instance, [14, Theorem 5.4, p. 40]). It is relevant to point out that a generalization of the Lipschitzian mappings, namely, the $\mathcal{D}$-Lipschitzian mappings (see Definition 3.1), are frequently used to prove the existence of T (see [8] and references therein).

Note that in Darbo fixed point theorem MNCs need not satisfy monotony property (iii). It is enough to suppose property $(iii)'$. Indeed, as we have indicated in Remark 1.1 by defining $\mu(S) := \mathrm{Diam}(S)$, Darbo fixed point theorem follows.

In this paper we present some new fixed point results that are not based on MNCs. Our main tool is the so-called *degree of nondensifiability* (briefly, DND) based on α*-dense curves* (a generalization of the *space-filling curves)*. Both concepts are explained in detail in Section 2. In [16] the DND has been applied to prove the existence of solutions of certain integral equations of fractional order. To attain our goal, we will first prove, using the DND, a fixed point result that works out under more general conditions than Darbo fixed point theorem and its known generalizations.

Following the above research line, we have introduced in Section 3.1 a new class of mappings, larger than those of $\mathcal{D}$-Lipschitzian. Furthermore, in Section 3.2 we have proved a fixed point result in Banach algebras without using Darbo fixed point theorem for χ. As we will show in several examples, the hypothesis of Darbo fixed point theorem and its generalizations are not fulfilled.

The usefulness of our results are evidenced in Section 4 where we have demonstrated the existence of solutions for certain quadratic integral equations. We have also prove that the sufficient conditions of Proposition 4.1 are more general than others required in some of the above cited works.

2. The degree of nondensifiability

Let (E, d) be a metric space and $\mathfrak{B}_E$ the class of non-empty and bounded subsets of E. In 1997 the concept of α-dense curve was introduced by Mora and Cherruault [17]:

Definition 2.1. *Let $\alpha \geq 0$ and $D \in \mathfrak{B}_E$. A continuous mapping $\gamma : I \longrightarrow (E, d)$ is said to be an α-dense curve in D if it satisfies the two conditions:*

(i) $\gamma(I) \subset D$.
(ii) *For any $x \in D$, there is $y \in \gamma(I)$ such that $d(x, y) \leq \alpha$.*

Note that, given $D \in \mathfrak{B}_E$, there is an α-dense curve in D for any $\alpha \geq \mathrm{Diam}(D)$, the diameter of D. Indeed, fixed a point $x_0 \in D$, the mapping $\gamma(t) := x_0$ for all $t \in I$ is an α-dense curve in D provided that $\alpha \geq \mathrm{Diam}(D)$. If D is a connected, compact and locally connected set, by Hahn-Mazurkiewicz theorem (see [18]), there exists a continuous mapping γ such that $\gamma(I) = D$ and then γ is called a space-filling curve. Since γ obviously satisfies the conditions of Definition 2.1 for $\alpha = 0$, in particular, γ is an α-dense curve in D. Therefore the α-dense curves generalize the space-filling curves.

The α-dense curves generate a class of sets in (E, d) called densifiable sets.

Definition 2.2. *A set $D \in \mathfrak{B}_E$ is said to be densifiable if for every $\alpha > 0$ there is an α-dense curve in D.*

The class of densifiable sets is strictly between the class of Peano Continua (the sets that are a continuous image of I) and the class of connected and precompact sets (see [19]). For a detailed exposition of the above concepts, see [17, 19, 20, 21] and references therein.

The notion of DND is obtained from the above concept of α-dense curve.

Definition 2.3. *The degree of nondensifiability (DND) is the mapping* $\phi_d : \mathfrak{B}_E \longrightarrow \mathbb{R}_+$ *defined as*

$$\phi_d(D) := \inf\{\alpha \geq 0 : \Gamma_{\alpha,D} \neq \emptyset\}, \quad D \in \mathfrak{B}_E,$$

$\Gamma_{\alpha,D}$ *being the class of* α*-dense curves in* D.

As we have pointed out above, $\Gamma_{\alpha,D} \neq \emptyset$ for any $\alpha \geq \mathrm{Diam}(D)$, so ϕ_d is well defined. In accord with the dimension of X, we have $\phi_d(B_X) = 0$ (see Proposition 2.1 below) if X is finite dimensional, and $\phi_d(B_X) = 1$ if X has infinite dimension (see [21]).

Example 2.1. *Let* L^1 *be the Banach space of absolute value Lebesgue integrable functions defined on* I *endowed its usual norm. For the set of the statistics density functions*

$$D := \Big\{f \in L^1 : f \geq 0 \text{ and } \int_0^1 f(x)dx = 1\Big\},$$

one has $\phi_d(D) = 2$ *(see [15]). Therefore the inequality*

$$1 = \phi_d(B_{L^1}) = \phi_d(B_{L^1} \cup D) < \max\{\phi_d(D), \phi_d(B_{L^1})\} = 2,$$

means that the DND ϕ_d *is not a MNC because the monotony condition of Definition 1.1 is not satisfied.*

From now on, we will assume that $(X, \|\cdot\|)$ is a Banach algebra satisfying the condition $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in X$.

In spite of ϕ_d is not a MNC, it has properties very close to it.

Proposition 2.1. *The DND satisfies the following properties:*

(1) *Regularity on the subfamily* $\mathfrak{B}_{a,X} \subset \mathfrak{B}_X$ *of arc-connected sets:* $\phi_d(S) = 0$ *if and only if* S *is a precompact set, for each* $S \in \mathfrak{B}_{a,X}$.
(2) *Invariant under closure:* $\phi_d(S) = \phi_d(\bar{S})$, *for each* $S \in \mathfrak{B}_X$.
(3) *Semi-homogeneity:* $\phi_d(\lambda S) = |\lambda|\phi_d(S)$, *for each* $\lambda \in \mathbb{R}$ *and* $S \in \mathfrak{B}_X$.
(4) *Invariant under translations:* $\phi_d(x + S) = \phi_d(S)$, *for each* $x \in X$ *and* $S \in \mathfrak{B}_X$.
(5) $\phi_d(\mathrm{Conv}(S_1)) \leq \phi_d(S_1)$ *and*

$$\phi_d(\mathrm{Conv}(S_1 \cup S_2)) \leq \max\{\phi_d(\mathrm{Conv}(S_1)), \phi_d(\mathrm{Conv}(S_2))\},$$

for each $S_1, S_2 \in \mathfrak{B}_X$.
(6) $\phi_d(S_1 + S_2) \leq \phi_d(S_1) + \phi_d(S_2)$, *for each* $S_1, S_2 \in \mathfrak{B}_X$.
(7) $\phi_d(S_1 S_2) \leq \phi_d(S_1)\|S_2\| + \phi_d(S_2)\|S_1\| + \phi_d(S_1)\phi_d(S_2)$, *for each* $S_1, S_2 \in \mathfrak{B}_X$, *where* $\|S_i\| := \sup\{\|x\| : x \in S_i\}$, *for* $i = 1, 2$.

Proof. We only prove (6) and (7) (for a proof of (1)-(5) see [15]). Consider α_i-dense curves in S_i, $\gamma_i : I \longrightarrow X$ with $\alpha_i > \phi_d(S_i)$, $i = 1, 2$ and let $\tau : I \longrightarrow I^2$, $\tau(t) := (\tau_1(t), \tau_2(t))$, $t \in I$, a space-filling curve in I^2 (see [18]). Then given $x_i \in S_i$ there are $t_i \in I$ such that

$$\|x_i - \gamma_i(t_i)\| \leq \alpha_i, \quad i = 1, 2.$$

Now by defining $\tilde{\gamma}(t) := \gamma_1(\tau_1(t)) + \gamma_2(\tau_2(t))$, the above inequalities imply that $\tilde{\gamma}$ is an $(\alpha_1 + \alpha_2)$-dense curve in $S_1 + S_2$. Thus, in view of the arbitrariness of $\alpha_i > \phi_d(S_i)$, for $i = 1, 2$, we conclude that $\phi_d(S_1 + S_2) \leq \phi_d(S_1) + \phi_d(S_2)$, which proves (6).

To demonstrate (7), first, let us note that the condition (ii) of Definition 2.1 is equivalent to say

$$S_i \subset \gamma_i(I) + \alpha_i B_X, \quad \text{for } i = 1, 2 \tag{2.1}$$

Therefore given $z := x_1 x_2 \in S_1 S_2$ with $x_i \in S_i$, $i = 1, 2$, from (2.1) we can express $z = (y_1 + \alpha_1 u_1)(y_2 + \alpha_2 u_2)$ for some $y_i \in \gamma_i(I)$ and $u_i \in B_X$, $i = 1, 2$. By a simple verification we then deduce

$$\|z - y_1 y_2\| \leq \alpha_1 \|y_2\| + \alpha_2 \|y_1\| + \alpha_1 \alpha_2. \tag{2.2}$$

Now, define $\omega : I \longrightarrow X$ as $\omega(t) := \gamma_1(\tau_1(t))\gamma_2(\tau_2(t))$, $t \in I$. Clearly, ω is continuous and $\omega(I) \subset S_1 S_2$. By taking $t_i \in I$ such that $\gamma_i(\tau_i(t_i)) = y_i$, $i = 1, 2$, from (2.2) we deduce that ω is an α-dense curve in $S_1 S_2$ for $\alpha := \alpha_1 \|S_2\| + \alpha_2 \|S_1\| + \alpha_1 \alpha_2$. Therefore $\phi_d(S_1 S_2) \leq \alpha$. Finally, noticing the arbitrariness of $\alpha_i > \phi_d(B_i)$, for $i = 1, 2$, the property (7) follows.

□

Note that the properties (2)-(4) and (6) of the above result remain true if we replace the DND ϕ_d by the Hausdorff MNC χ (see, for instance, [14] and also [8, Lemma 2.4]).

In the subclass $\mathfrak{B}_{a,X}$ of arc-connected sets of $\mathfrak{B}_X$, we have the following inequalities (see [15, Theorem 2.5]):

Proposition 2.2. *The inequalities*

$$\chi(S) \leq \phi_d(S) \leq 2\chi(S),\ S \in \mathfrak{B}_{a,X},$$

are the best possible in infinite dimensional Banach spaces.

Now, we focus in the particular case of the sets of the form

$$\{A(x)B(x) + C(x) : x \in \Omega\},$$

where A, B, $C : \Omega \longrightarrow X$ are continuous, X is a Banach algebra of functions and Ω is a convex set of $\mathfrak{B}_X$. Assume $A(\Omega)$, $B(\Omega)$, $C(\Omega) \in \mathfrak{B}_X$. Under these conditions, the properties (6) and (7) of Proposition 2.1 remain true for a MNC μ. In particular, from the properties of the Hausdorff MNC χ and the condition

$\mu(B_1) \leq \mu(B_2)$, for $B_1 \subset B_2 \subset \Omega$ (equivalent to condition (iii) of Definition 1.1), we infer

$$\begin{aligned} &\chi(\{A(x)B(x) + C(x) : x \in \Omega\}) \leq \chi(A(\Omega)B(\Omega) + C(\Omega)) \leq \\ &\leq \chi(A(\Omega))\|B\| + \chi(B(\Omega))\|A\| + \chi(A(\Omega))\chi(B(\Omega)) + \chi(C(\Omega)). \end{aligned} \tag{2.3}$$

Since ϕ_d has not the *monotony* property (see Example 2.1), the inequality (2.3) may not be satisfied by ϕ_d. However, in view of (2.3) and Proposition 2.2, the DND fulfills:

$$\begin{aligned} &\phi_d(\{A(x)B(x) + C(x) : x \in \Omega\}) \leq \\ &\leq 2\big[\phi_d(A(\Omega))\|B\| + \phi_d(B(\Omega))\|A\| + \phi_d(A(\Omega))\phi_d(B(\Omega)) + \phi_d(C(\Omega))\big]. \end{aligned} \tag{2.4}$$

We will use this inequality later.

3. Main results

For clarity, we divide this section in two subsections. In the first one we define the key concepts and in the second we prove the fixed points results that will be applied in the analysis of certain quadratic integral equations in Section 4.

3.1. Preliminary definitions and results

To setting the notation, let

$$\mathcal{D} := \{h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ : h \text{ is continuous nondecreasing and } h(0) = 0\}.$$

The following concepts, due to Dhage [22], are crucial for our goal.

Definition 3.1. *A mapping $T : \Omega \subseteq X \longrightarrow X$ is said to be $\mathcal{D}$-Lipschitzian if there is $h \in \mathcal{D}$ such that $\|T(x) - T(y)\| \leq h(\|x - y\|)$ for each x, $y \in \Omega$. The function h is then called a $\mathcal{D}$-function of T. If, in addition, h satisfies $h(r) < r$ for all $r > 0$, then T is called a $\mathcal{D}$-nonlinear contraction with a contraction function h.*

Remark 3.1. *Every Lipschitzian mapping is a $\mathcal{D}$-Lipschitzian mapping, but the converse is not true in general. Indeed, we can take the mapping $T : \mathbb{R} \longrightarrow \mathbb{R}$ defined as $T(x) := \sqrt{|x|}$ which is not Lipschitzian but it is $\mathcal{D}$-Lipschitzian with $h(r) := \sqrt{r}$ as $\mathcal{D}$-function; see [8, Remark 2.1].*

In [8, Lemma 2.1] it is shown that if $T : X \longrightarrow X$ is a $\mathcal{D}$-Lipschitzian mapping with a $\mathcal{D}$-function h, and $T(S) \in \mathfrak{B}_X$ for each $S \in \mathfrak{B}_X$, then

$$\chi(T(S)) \leq h(\chi(S)), \text{ for all } S \in \mathfrak{B}_X. \tag{3.1}$$

In the next result we prove that the above property is also true for ϕ_d, even if T is defined on a subset of X.

Proposition 3.1. *Let $\Omega \subseteq X$ non-empty and $T : \Omega \longrightarrow X$ be a $\mathcal{D}$-Lipschitzian mapping with a $\mathcal{D}$-function h such that $T(\Omega) \in \mathfrak{B}_X$. Then*

$$\phi_d(T(S)) \leq h(\phi_d(S)),$$

for each $S \subset \Omega$ non-empty and bounded.

Proof. Given a non-empty and bounded set $S \subset \Omega$ and $\alpha > \phi_d(S)$, let γ be an α-dense curve in S. Then, given $x \in S$ there is $t \in I$ such that

$$\|x - \gamma(t)\| \leq \alpha. \tag{3.2}$$

Clearly, the mapping $T \circ \gamma : I \longrightarrow X$ is continuous and $T(\gamma(I)) \subset T(S)$. Given $y \in T(S)$, so $y := T(x)$ for some $x \in S$, noticing (3.2) and the properties of T there is $t \in I$ such that

$$\|y - T(\gamma(t))\| \leq h(\|x - \gamma(t)\|) \leq h(\alpha),$$

which means that $\phi_d(T(S)) \leq h(\alpha)$. Now, by tending $\alpha \to \phi_d(S)$, the proposition follows from the above inequality and the continuity of h. $\square$

It is important to stress that there are spaces for which, on certain $\Omega \in \mathfrak{B}_X$, it may be defined an isometry $T : \Omega \longrightarrow \Omega$ such that T is a $\mathcal{D}$-Lipschitzian mapping with a $\mathcal{D}$-function h equal to the identity in such a way that (3.1) fails (see [23]).

By virtue of (3.1) and Proposition 2.2, noticing h is nondecreasing, the next result follows immediately for the Hausdorff MNC χ.

Corollary 3.1. *Let $\Omega \subseteq X$ non-empty and $T : \Omega \longrightarrow X$ be a $\mathcal{D}$-Lipschitzian mapping with a $\mathcal{D}$-function h such that $T(\Omega) \in \mathfrak{B}_X$. Then*

$$\chi(T(S)) \leq h(2\chi(S)),$$

for each $S \subset \Omega$ non-empty and bounded.

Remark 3.2. *The above inequality is achieved, see [23].*

In view of the above results, it is convenient to introduce the following classes of mappings.

Definition 3.2. *Let $\Omega \in \mathfrak{B}_X$ convex and $T : \Omega \longrightarrow X$ a mapping such that $T(\Omega) \in \mathfrak{B}_X$. We will say that T is $(\phi_d, \mathcal{D})$-Lipschitzian with a $\mathcal{D}$-function h if for each non-empty and convex set $S \subset \Omega$,*

$$\phi_d(T(S)) \leq h(\phi_d(S)).$$

If this inequality holds for a contraction function $h \in \mathcal{D}$ and each non-empty and convex set $S \subset \Omega$ with $\phi_d(S) > 0$, then T is said to be a $(\phi_d, \mathcal{D})$-nonlinear contraction with a contraction function $h \in \mathcal{D}$.

Let us note that the class of $(\phi_d, \mathcal{D})$-nonlinear contractions is larger than the $\mathcal{D}$-nonlinear contractions one. For instance, a compact mapping (see, for instance, [14, Definition 2.5, p. 13]), in view of Proposition 2.1, is a $(\phi_d, \mathcal{D})$-nonlinear contraction for any contraction function $h \in \mathcal{D}$. Of course, a compact mapping is not necessarily a $\mathcal{D}$-nonlinear contraction. The same can be stated for the class of $(\phi_d, \mathcal{D})$-Lipschitzian and $\mathcal{D}$-Lipschitzian mappings. A less trivial example is the following:

Example 3.1. *Let $(\ell_2, \|\cdot\|_2)$ be the Banach space of the real sequences with $\|x\|_2 := (\sum_{n\geq 1} |x_n|^2)^{1/2} < +\infty$, for $x := (x_1, x_2, \ldots, x_n, \ldots)$. Fixed $0 < \beta < 1/2$, on the closed unit ball B_{ℓ_2} we define $T : B_{\ell_2} \longrightarrow B_{\ell_2}$ as*

$$T(x) := (\sqrt{1 - \|x\|_2}, \beta x_1, \beta x_2, \ldots, \beta x_n, \ldots),\ x \in B_{\ell_2}.$$

Then, if $\|x\|_2 = 1$ we have $\|T(x) - T(\theta)\|_2 = \sqrt{1 + \beta^2} > 1 = \|x - \theta\|_2$, θ being the null vector of ℓ_2. Therefore T can not be a k-contraction for any $0 \leq k < 1$.

Given a non-empty and convex set $S \subset B_{\ell_2}$, it can be shown (see [14, Example 7, p. 41]) that $\chi(T(S)) \leq \beta\chi(S)$ and therefore, noticing Proposition 2.2,

$$\phi_d(T(S)) \leq 2\beta\chi(S) \leq 2\beta\phi_d(S).$$

Consequently T is a $(\phi_d, \mathcal{D})$-nonlinear contraction with a contraction function $h(r) := 2\beta r$.

3.2. Fixed points of $(\phi_d, \mathcal{D})$-Lipschitzian mappings

The next result is decisive for our goal:

Theorem 3.1. *Let $\Omega \in \mathfrak{B}_X$ be a closed and convex set, and $T : \Omega \longrightarrow \Omega$ a $(\phi_d, \mathcal{D})$-nonlinear contraction with a contraction function $h \in \mathcal{D}$. Then T has some fixed point.*

Proof. Fixed $x_0 \in \Omega$ we define the class

$$\mathfrak{C} := \Big\{ S \subset \Omega : S \text{ is non-empty closed convex, } x_0 \in S, T(S) \subset S \Big\}.$$

Note that $\mathfrak{C}$ is non-empty because $\Omega \in \mathfrak{C}$. Define the sets

$$F := \bigcap_{S \in \mathfrak{C}} S, \quad G := \overline{\mathrm{Conv}}\Big(T(F) \cup \{x_0\}\Big).$$

Since $x_0 \in F$, we have $F \neq \emptyset$. Note that $x_0 \in S$ and $T(F) \subset F$, so $G \subset F$ and then $T(G) \subset T(F) \subset G$. That is, $G \in \mathfrak{C}$, and consequently $F \subset G$. Then it follows that $F = G$.

Now we claim that F is precompact. Otherwise, since F is convex, so arc-connected, by Proposition 2.1 we have $\phi_d(F) > 0$ and then again by Proposition

2.1 and using Definitions 3.1 and 3.2, we infer

$$\begin{aligned}\phi_d(F) &= \phi_d\Big(\overline{\mathrm{Conv}}(T(F)\cup\{x_0\})\Big) = \phi_d\Big(\mathrm{Conv}(T(F)\cup\{x_0\})\Big)\\ &\leq \max\Big\{\phi_d(\mathrm{Conv}(T(F))), \phi_d(\{x_0\})\Big\} = \phi_d(\mathrm{Conv}(T(F))) \leq \phi_d(T(F))\\ &\leq h(\phi_d(F)) < \phi_d(F),\end{aligned}$$

which is a contradiction. Therefore F is precompact as claimed. Since F is closed and convex, F is a convex compact subset of X and then by Schauder fixed point theorem, T has a fixed point.

□

Let us note that for the particular case that $h(r) := kr$, for some $0 \leq k < 1$, the above result becomes into the celebrated Darbo fixed point (see, for instance, [14]) replacing the DND ϕ_d by a MNC invariant under convex hull. Nevertheless, even under these conditions, the above result and Darbo fixed point theorem are essentially different. We evidence the difference between Darbo fixed theorem and Theorem 3.1 by means of the next example due to Akhmerov *et al* [13].

Example 3.2. *Consider the Banach space $\mathcal{C}(I)$ of the continuous functions defined on I, endowed the usual supremum norm $\|\cdot\|_\infty$, and let $(p_n)_{n\geq 1}$, $(q_n)_{n\geq 1}$, $(r_n)_{n\geq 1}$ and $(s_n)_{n\geq 1}$ be sequences in I such that $0 < s_{n+1} < p_n < q_n < r_n < s_n \to 0$. For each integer $n \geq 1$, define*

$$f_n(t) := \begin{cases} -1, & \text{for } t = r_n, s_n, 1\\ 1, & \text{for } t = 0,\, p_n,\, q_n \end{cases}, \quad g_n(t) := \begin{cases} -1, & \text{for } t = r_n\\ 0, & \text{for } t = 0, p_n, s_n, 1\\ 1, & \text{for } t = q_n \end{cases}$$

and extend f_n and g_n, by linear interpolation, to the whole interval I. Then, for each integers $n \neq m$, we have

$$\|f_n - f_m\|_\infty = 2, \quad \|g_n - f_m\|_\infty = 2, \quad \|g_n - g_m\|_\infty = 1. \tag{3.3}$$

Now, let $C := \mathrm{Conv}\{\theta, f_1, g_1, f_2, g_2, \ldots\}$, where θ is the identically null function, and $T : C \longrightarrow C$ defined as

$$T(\theta) = \theta, \quad T(f_n) = g_{2n-1}, \quad T(g_n) = g_{2n},$$

$$T(\lambda_1 h_1 + \ldots + \lambda_n h_n) = \lambda_1 T(h_1) + \ldots + \lambda_n T(h_n),$$

where $\lambda_i \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$, and $h_1, \ldots, h_n \in \{f_1, g_1, f_2, g_2, \ldots\}$. It is not difficult to check that T is well defined and, by (3.3), it satisfies

$$\|T(f) - T(g)\|_\infty = \frac{1}{2}\|f - g\|_\infty, \quad \text{for all } f,\, g \in C. \tag{3.4}$$

Then there is a unique extension of T, noted in the same way, $T : \overline{C} \longrightarrow \overline{C}$, that satisfies (3.4). Therefore T is a $(\phi_d, \mathcal{D})$-nonlinear contraction with a contraction function $h(r) := r/2$.

However one has (see [13, Remark 1.5.3, p. 22])

$$\chi\big(\{f_1, f_2, \ldots\}\big) = 1 = \chi\big(T(\{f_1, f_2, \ldots\})\big),$$

and then Darbo fixed point theorem (for the Hausdorff MNC χ) can not be applied here.

The celebrated paper of Krasnosel'skiĭ [24] states that if $\Omega \in \mathfrak{B}_X$ is closed and convex, $A, C : \Omega \longrightarrow X$ continuous and the conditions:

(1) $A(x) + C(x) \in \Omega$, for each $x \in \Omega$.
(2) A is a k-contraction, for some $k \in [0, 1)$, i.e. $\|A(x) - A(x)\| \le k\|x - y\|$ for all x, $y \in \Omega$.
(3) C is compact.

hold, then the equation $A(x) + C(x) = x$, $x \in \Omega$, has solution.

In our context, an analogous to the result of Krasnosel'skiĭ is the following:

Corollary 3.2. *Let $\Omega \in \mathfrak{B}_X$ be a closed and convex set and $A, C : \Omega \longrightarrow X$ continuous. Assume the conditions:*

(1) *$A(x) + C(x) \in \Omega$, for each $x \in \Omega$.*
(2) *A is $(\phi_d, \mathcal{D})$-Lipschitzian with a $\mathcal{D}$-function h_A.*
(3) *C is compact.*

In addition, suppose that

$$2h_A(\phi_d(S)) < \phi_d(S), \tag{3.5}$$

for each non-empty and convex set $S \subset \Omega$ with $\phi_d(S) > 0$. Then, the equation $A(x) + C(x) = x$ has a solution for some $x \in \Omega$.

Proof. The mapping $T := A + C$ is continuous and, by condition (1), satisfies $T(\Omega) \subset \Omega$. Let $S \subset \Omega$ be a non-empty and convex set with $\phi_d(S) > 0$. By virtue of Proposition 2.2 and in view of conditions (2) and (3) of the statement, we have

$$\phi_d(T(S)) \le 2(\chi(A(S)) + \chi(C(S))) = 2\chi(A(S)) \le 2\phi_d(A(S)) \le 2h_A(\phi_d(S)).$$

Thus, noticing (3.5), T is a $(\phi_d, \mathcal{D})$-nonlinear contraction and then the corollary follows by Theorem 3.1.

□

By Definition 3.2, a k-contraction $T : \Omega \longrightarrow X$, for some $0 \le k < 1$, is a $(\phi_d, \mathcal{D})$-nonlinear contraction with a contraction function $h(r) := kr$. However, the condition $h(r) < r/2$ required in (3.5), for all $r \in (0, \mathrm{Diam}(\Omega)]$, *a priori* may not be necessarily satisfied. Therefore to find an application of Corollary 3.2 where Krasnosel'skiĭ result does not work, we need to find a mapping that is not a k-contraction satisfying the inequality of such corollary. The following example illustrates this fact.

Example 3.3. *In* $X := \mathcal{C}(I)$ *we define the set*

$$\Omega := \big\{x \in X : 0 \le x(t) \le 1, \text{ for each } t \in I\big\},$$

which is clearly closed, and convex but not compact. Also, $\mathrm{Diam}(\Omega) = 1$ *and then* $0 \le \phi_d(S) \le 1$ *for every non-empty and convex set* $S \subset \Omega$.

Define the "sawtooth" periodic function $\xi : \mathbb{R} \longrightarrow \mathbb{R}$ *as*

$$\xi(t) := \begin{cases} |t|, & \text{for } t \in [-1,1] \\ \xi(t+2), & \text{otherwise} \end{cases},$$

and the function $h(t) := \sum_{k\ge 1} 2^{-k}\xi(2^k t)$, *for each* $t \in I$. *As* h *is nowhere differentiable (see, [25, Chap. V]), by Rademacher's theorem (see, for instance, [26, §3.1.6, p. 216])* h *is not Lipschitzian. However,* h *has the following property:* $|h(t) - h(t')| \le \alpha$ *whenever* $|t - t'| \le \alpha$, *for each* $\alpha > 0$.

On the other hand, fixed $0 < \beta < 1/2$ *and* $x_0 \in \Omega$ *let* $A, C : \Omega \longrightarrow X$ *the mappings given by*

$$A(x)(t) := \beta h\big(x(t)\big), \quad C(x)(t) := (1-\beta)x_0(t),$$

for each $x \in \Omega$ *and* $t \in I$.

Then, C *is compact but* A *is not Lipschitzian and, in particular, is not a* k*-contraction for any* $k \in [0,1)$. *Therefore the above exposed Krasnosel'skiĭ result can not be applied here. But, from the above considerations,* A *is a* $(\phi_d, \mathcal{D})$*-nonlinear contraction with a contraction function* $h_A(r) := \beta r$, *and so* $h_A(\phi_d(S)) \le \beta\phi_d(S) < \phi_d(S)/2$ *for each non-empty and convex set* $S \subset \Omega$ *with* $\phi_d(S) > 0$.

On the other hand, we have the following fixed point result in Banach algebras.

Theorem 3.2. *Let* $\Omega \in \mathfrak{B}_X$ *be a closed and convex set and* $A, B, C : \Omega \longrightarrow X$ *continuous. Assume the conditions:*

(1) $A(x)B(x) + C(x) \in \Omega$, *for each* $x \in \Omega$.
(2) A, C *are* $(\phi_d, \mathcal{D})$*-Lipschitzian with* $\mathcal{D}$*-functions* h_A *and* h_C, *respectively.*
(3) B *is compact.*

In addition, suppose that

$$2\big[\|B\| h_A(\phi_d(S)) + h_C(\phi_d(S))\big] < \phi_d(S), \tag{3.6}$$

for each non-empty and convex set $S \subset \Omega$ *with* $\phi_d(S) > 0$. *Then, the equation* $A(x)B(x) + C(x) = x$ *has some solution* $x \in \Omega$.

Proof. First, by the compactness of B, $\|B\|$ is well defined. Now, define $T : \Omega \longrightarrow X$ as $T(x) := A(x)B(x) + C(x)$, for each $x \in \Omega$. Clearly T is continuous and $T(\Omega) \subset \Omega$, from the condition (1). Bearing in mind the inequality (2.4), given a non-empty and convex set $S \subset \Omega$ with $\phi_d(S) > 0$, we have

$$\phi_d(T(S)) \le 2\big[\phi_d(A(S))\|B\| + \phi_d(B(S))\|A\| + \phi_d(A(S))\phi_d(B(S)) + \phi_d(C(S))\big]. \tag{3.7}$$

Now, as $\phi_d(B(S)) = 0$, from (1) of Proposition 2.1, and by condition (2) of the statement $\phi_d(Z(S)) \leq h_Z(\phi_d(S))$ for $Z := A$, C, in view of (3.7) we have

$$\phi_d(T(S)) \leq 2\big[\|B\|h_A(\phi_d(S)) + h_C(\phi_d(S))\big].$$

From (3.6), $\tilde{h} := 2[\|B\|h_A + h_C] \in \mathcal{D}$ is clearly a nonlinear contraction, so T is a $(\phi_d, \mathcal{D})$-nonlinear contraction. The result follows then by Theorem 3.1. □

Before continuing, some comments are needed.

(I) The mappings A and C are $(\phi_d, \mathcal{D})$-Lipschitzian instead $\mathcal{D}$-Lipschitzian as required in most works cited in Section 1. As we have shown in Section 3.1, this condition is more general than the $\mathcal{D}$-Lipschitzian condition. Furthermore, note that we do not need to prove the existence of the inverse of any mapping.

(II) The class of $(\phi_d, \mathcal{D})$-nonlinear contractions is larger than others classes of mappings used in certain generalizations of Darbo fixed point theorem. For instance, in [4, Theorem 3] (see also Remark 1.1), if $T : \Omega \longrightarrow \Omega$ is continuous, with $\Omega \in \mathfrak{B}_X$ convex and closed, the following condition is required
$$\mu(T(S)) \leq \psi(\mu(S)),$$
for each non-empty $S \subset \Omega$, where ψ is a nondecreasing function such that $\lim_n \psi^n(r) = 0$ for each $r \geq 0$, where the exponent means composition. As one can easily check, under those conditions, $\psi \in \mathcal{D}$ and it is a nonlinear contraction.

(III) Surely, a result similar to Theorem 3.2 for the Hausdorff MNC χ can be proved. But, in view of Corollary 3.1, the inequality (3.6) of the above theorem must be of the form

$$\|B\|h_A(2\chi(S)) + h_C(2\chi(S)) < \chi(S),$$

for each non-empty $S \subset \Omega$ with $\chi(S) > 0$. Of course, in general, the above inequality is more difficult to check than (3.6).

(IV) Note if we take the "inner" (also called "relative") Hausdorff MNC χ_i, see [13, 23], defined as

$$\chi_i(S) := \inf\big\{\varepsilon > 0 : S \text{ can be covered by finitely many balls} \\ \text{with centers in } S \text{ and radii } \leq \varepsilon\big\} \quad \text{for all } S \in \mathfrak{B}_X,$$

instead the DND ϕ_d in Examples 3.1, 3.2 and 3.3, the obtained results remain true. However, χ_i is not a MNC (see [13, p. 12]). Also, the DND ϕ_d and χ_i are essentially different mappings. Indeed, for

$$S := \big\{(x, \sin(1/x)), x \in [-1, 0) \cup (0, 1]\big\} \cup \big\{[0, y] : y \in [-1, 1]\big\} \subset \mathbb{R}^2$$

we find that $\chi_i(S) = 0 < 1 = \phi_d(S)$. See also Example 3.4 below.

A special case of Theorem 3.2 is when C is a compact mapping. Indeed, as $\phi_d(C(S)) = 0$ for each non-empty and convex $S \subset \Omega$, the inequality (3.6) can be replaced by $2\|B\|h_A(\phi_d(S)) < \phi_d(S)$ whenever $\phi_d(S) > 0$. Formally:

Corollary 3.3. *Let $\Omega \in \mathfrak{B}_X$ be closed and convex and $A, B, C : \Omega \longrightarrow X$ continuous. Assume the conditions:*

(1) *$A(x)B(x) + C(x) \in \Omega$ for each $x \in \Omega$.*
(2) *A is $(\phi_d, \mathcal{D})$-Lipschitzian with a $\mathcal{D}$-function h_A.*
(3) *B and C are compact.*

In addition, assume that

$$2\|B\|h_A(\phi_d(S)) < \phi_d(S),$$

for each non-empty and convex set $S \subset \Omega$ with $\phi_d(S) > 0$. Then the equation $A(x)B(x) + C(x) = x$ has some solution $x \in \Omega$.

We conclude this section with the following example.

Example 3.4. *Let $X := \mathcal{C}(I)$ and Ω as in Example 3.3. Fixed $x_0 \in \Omega$, define the mappings $A, B, C : \Omega \longrightarrow X$ as*

$$A(x)(t) := \log\big(1 + x(t)\big), \quad B(x)(t) := \frac{1}{2}, \quad C(x)(t) = \frac{1}{2}x_0(t),$$

for every $x \in \Omega$ and $t \in I$. Let us note that A is not a k-contraction, for any $0 \le k < 1$, but as $\log(1 + a) - \log(1 + b) \le \log(1 + |a - b|)$ for each $a, b > 0$, $a \ne b$, we can derive the inequality

$$\phi_d(A(S)) \le \log(1 + \phi_d(S)) < \phi_d(S),$$

for every non-empty and convex set $S \subset \Omega$ with $\phi_d(S) > 0$. So, the conditions of Corollary 3.3 are satisfied, taking $h_A(r) := \log(1 + r)$.

On the other hand, we have $\chi(\Omega) = 1/2$ and we prove in the following lines that $\chi(A(\Omega)) = 1/2$. Let us note that $|\log(1+x(t))-\log(1+1/2)| \le |x(t)-1/2| \le 1/2$, and so, $\chi(A(\Omega)) \le 1/2$.

If we can find $f_1, \dots, f_n \in X$ and $0 < r < 1/2$ such that

$$A(\Omega) \subset \bigcup_{i=1}^{n} B(f_i, r).$$

Then, for a given $\varepsilon > 0$, taking $x(t) := t^m \in \Omega$ there is $\delta > 0$ such that $t^m \le \varepsilon$ for each $t \in [1 - \delta, 1)$. Let f_i be such that $\|\log(1 + x) - f_i\|_\infty \le r$. If $g \in \Omega \cap B(f_i, r)$, we have

$$r \ge \|\log(1 + x) - f_i\|_\infty \ge \|x - g\|_\infty - \frac{1}{2} \ge g(t) - \log(1 + \varepsilon) - \frac{1}{2},$$

for each $t \in [1 - \delta, 1)$. Letting $t \to 1^-$, from the above inequality we have

$$r \ge \frac{1}{2} - \log(1 + \varepsilon),$$

which, taking into account the arbitrariness of ε*, is contradictory with the assumption that* $r < 1/2$*.*

Let us note that $\chi_i(\Omega) = 1$ *and* $\chi_i(A(\Omega)) \geq \chi(A(\Omega)) = 1/2$ *(see comment (IV) above), and therefore the inequality* $\chi_i(A(S)) \leq \log(1 + \chi_i(S))$ *for every* $S \subset \Omega$ *with* $\chi_i(S) > 0$ *does not hold.*

4. Existence of solutions for certain quadratic integral equations

Fix $T > 0$ and let X be the Banach algebra of continuous functions $x : [0, T] \longrightarrow \mathbb{R}$ equipped with the usual supremum norm $\|\cdot\|_\infty$. In infinite dimensional Banach spaces, a large class of compact mappings is that of integral operators with sufficiently regular kernels (see, for instance, [27]). In our next result, we will apply Corollary 3.3 to prove the existence of solutions for the quadratic integral equations of the form

$$x(t) = q(t) + \Big[f(t, x(t)) + \int_0^{p_1(t)} K_1(t, s, x(s))ds\Big] \int_0^{p_2(t)} K_2(t, s, x(s))ds, \quad (4.1)$$

for each $t \in [0, T]$, where $q : [0, T] \longrightarrow \mathbb{R}$, $f : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $p_i : [0, T] \longrightarrow [0, T]$, $K_i : [0, T]^2 \times \mathbb{R} \longrightarrow \mathbb{R}$ for $i = 1, 2$ are known.

Note that for $f \equiv 0$, $p_2 \equiv 1$, $K_2 \equiv 1$ and $p_1(t) := t$ we find the well known nonlinear Volterra integral equation

$$x(t) = q(t) + \int_0^t K_1(t, s, x(s))ds,$$

while if $f \equiv 0$, $p_2 \equiv 1$, $K_2 \equiv 1$ and $p_1(t) := 1$ we have the Urysohn integral equation. Likewise, equation (4.1) is more general than that was analysed in [4, 7, 8, 10]. In [2], the above equation is considered for $K_1(t, s, x(s)) := \sigma(t, s)x(s)$, $K_2(t, s, x(s)) := \mu(s, t)K(s, x(s))$ and $T = p_1 = p_2 := \infty$, for certain functions σ, μ, K. As we will see in Example 4.2, equation (4.1) contains the celebrated integral equation of Ambartsumian-Chandrasekhar type.

Assume the following conditions are satisfied.

(C1) The functions q, f and p_i, K_i, for $i = 1, 2$, are continuous.

(C2) There are functions ψ, ψ_1, $\psi_2 : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$|f(t, x(t))| \leq \psi(R), \qquad \Big|\int_0^{p_i(t)} K_i(t, s, x(s))ds\Big| \leq \psi_i(R), \quad i = 1, 2,$$

whenever $\|x\|_\infty \leq R$, for each $x \in X$. Likewise, there is $R_0 > 0$ such that

$$\frac{\max\{|q(t)| : t \in [0, T]\}}{R_0} + \Big(\frac{\psi(R_0)}{R_0} + \frac{\psi_1(R_0)}{R_0}\Big)\frac{\psi_2(R_0)}{R_0} \leq 1. \quad (4.2)$$

(C3) Let $\Omega := \{x \in X : \|x\|_\infty \leq R_0\}$, with R_0 as above, and assume there is $h \in \mathcal{D}$ such that for every non-empty and convex set $S \subset \Omega$,

$$\phi_d\big(\{f(\cdot, x(\cdot)) : x \in S\}\big) \leq \beta h(\phi_d(S)) \quad \text{for some } \beta > 0.$$

Moreover,

$$\beta \sup\Big\{\Big|\int_0^{p_2(t)} K_2(t,s,x(s))ds\Big| : x \in S\Big\} h(\phi_d(S)) < \frac{\phi_d(S)}{2}, \tag{4.3}$$

for every $t \in [0,T]$, whenever $\phi_d(S) > 0$.

We can now state and prove the following result:

Proposition 4.1. *Assume conditions (C1)-(C3). Then, the equation (4.1) has some solution $x \in \Omega$, where Ω has been defined in condition (C3).*

Proof. Define the mappings A, B, $C : X \longrightarrow X$ as

$$A(x)(t) := f(t,x(t)), \quad B(x)(t) := \int_0^{p_1(t)} K_1(t,s,x(s))ds,$$

$$C(x)(t) := q(t) + \int_0^{p_1(t)} K_1(t,s,x(s))ds \int_0^{p_2(t)} K_2(t,s,x(s))ds \quad \forall t \in [0,T],$$

for each $x \in X$. Let us note that the above mappings are well defined and are continuous by condition (C1). Since a solution for the equation (4.1) is equivalent to the existence of some fixed point of $A(x)B(x) + C(x)$, we will apply Corollary 3.3 to show the existence of such fixed point in the set $\Omega := \{x \in X : \|x\|_\infty \le R_0\}$, where $R_0 > 0$ is given in condition (C2).

For clarity, we will divide the remain of the proof into several steps.

Step 1. $A(x)B(x) + C(x) \in \Omega$ for each $x \in \Omega$.

Let $M := \max\{|q(t)| : t \in [0,T]\}$. From (4.2), fixed $t \in [0,T]$, for each $x \in \Omega$ we have

$$\frac{|A(x)(t)B(x)(t) + C(x)(t)|}{R_0} \le \frac{|A(x)(t)||B(x)(t)| + |C(x)(t)|}{R_0}$$

$$\le \frac{M}{R_0} + \Big(\frac{\psi(R_0)}{R_0} + \frac{\psi_1(R_0)}{R_0}\Big)\frac{\psi_2(R_0)}{R_0} \le 1,$$

and from the arbitrariness of $t \in [0,T]$, $\|A(x)(t)B(x)(t) + C(x)(t)\|_\infty \le R_0$. So, $A(x)B(x) + C(x) \in \Omega$ for each $x \in \Omega$.

Step 2. B and C are compact mappings. This fact immediately follows taking into account that the mapping

$$x \in \Omega \longmapsto \int_0^{p(t)} K(t,s,x(s))ds$$

is compact for $p := p_1$, p_2, $K := K_1$, K_2 (courtesy of Arzelá-Ascoli theorem, see also [14, Example 3, p. 13]).

Step 3. A is a $(\phi_d, \mathcal{D})$-Lipschitzian mapping.

Indeed, from condition (C3), given a non-empty and convex set $S \subset \Omega$,

$$\phi_d(A(S)) = \phi_d(\{f(\cdot, x(\cdot)) : x \in S\}) \le \beta h(\phi_d(S)).$$

Then, since $\beta h \in \mathcal{D}$, A is $(\phi_d, \mathcal{D})$-Lipschitzian.

Step 4. Relationship between $2\|B\|\phi_d(A(S))$ and $\phi_d(S)$.

Let $S \subset \Omega$ be a non-empty and convex set with $\phi_d(S) > 0$. Recalling that $\|B\| := \sup\{\|B(x)\|_\infty : x \in \Omega\}$, in view of condition (C3) and the inequality (4.3), one has

$$2\|B\|\phi_d(A(S)) \le 2\beta\|B\|h(\phi_d(S)) < \phi_d(S).$$

Therefore the assumptions of Corollary 3.3 are satisfied and then the proof is now complete.

$\square$

Example 4.1. *Consider the integral equation*

$$x(t) := t^2 + \Big[\frac{\ln(1+|x(t)|)}{1+t} + \int_0^t \frac{x(t)\sin(2\pi x(t))}{1+st}ds\Big]\int_0^{\frac{t}{2}} e^{-sx^2(s)}ds, \quad \textit{for all } t \in I. \tag{4.4}$$

It is clear that all involved functions are continuous and

$$\Big|\frac{\ln(1+|x(t)|)}{1+t}\Big| \le R, \quad \Big|\int_0^t \frac{x(t)\sin(2\pi x(t))}{1+st}ds\Big| \le R, \quad \Big|\int_0^{\frac{t}{2}} e^{-sx^2(s)}ds\Big| \le \frac{1}{2},$$

whenever $\|x\|_\infty \le R$. Moreover

$$\frac{\max\{t^2 : t \in I\}}{R} + \Big(\frac{R}{R} + \frac{R}{R}\Big)\frac{1}{2R} = \frac{2}{R},$$

so the conditions (C1) and (C2) hold for every $R_0 \ge 2$. In the following lines we will show that condition (C3) is satisfied.

Fixed $R_0 \ge 2$, let $\Omega := \{x \in X : \|x\|_\infty \le R_0\}$ and $S \subset \Omega$ a non-empty and convex set with $\phi_d(S) > 0$. By the properties of the logarithm function we have

$$\phi_d\big(\{f(\cdot, x(\cdot)) : x \in S\}\big) \le \ln(1 + \phi_d(S)),$$

where $f(t, x(t)) := \ln(1 + |x(t)|)/(1 + t)$. Likewise, following the notation of condition (C3), by taking $\beta := 1$, $h(r) := \ln(1 + r)$, we deduce:

$$\sup\Big\{\Big|\int_0^{\frac{t}{2}} e^{-sx^2(s)}ds : x \in S\Big\}\ln(1 + \phi_d(S)) < \frac{\phi_d(S)}{2}.$$

Therefore, by Proposition 4.1, the equation (4.4) has some solution $x \in \Omega$.

It is important to stress that conditions (C1)-(C3) are more general than those proposed in others works. For instance, in [4] the following conditions for the function $f(t, x)$ are required:

(1) $|f(t, x) - f(t, y)| \le h(|x - y|)$, for every $t \in \mathbb{R}_+$ and x, $y \in \mathbb{R}$.

(2) $h(r) + h(s) \le h(r + s)$, for every $r, s \in \mathbb{R}_+$.

where $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is an upper semicontinuous function with $\lim_n h^n(r) = 0$, for every $r \in \mathbb{R}_+$. Therefore, in the above example, the function $h(r) := \ln(1+r)$ obeys condition (1) but not (2).

Likewise, in [5, 6] Lipschitzian conditions for the involved functions are required. Clearly, the boundness conditions exposed in condition (C2) are fulfilled for Lipschitzian functions, but the reciprocal does not hold in general. Indeed, one can think, for instance, in the square root function.

To end the paper, we will show in the following example the existence of continuous solutions for the so called Ambartsumian-Chandrasekhar integral equation, which is used in the theory of radiative transfer in semi-infinite atmospheres (see, for instance, [2, 28, 29]).

Example 4.2. *Let the Ambartsumian-Chandrasekhar integral equation*

$$x(t) = q(t) + x(t) \int_0^1 \frac{t}{t+s} x(s)ds \quad \textit{for all } t \in I, \tag{4.5}$$

where $q(t)$ is a known nonnegative continuous function such that $\int_0^1 q(t)dt \leq 1/2$. As it is pointed out in [29], equation 4.5 has a unique nonnegative solution $x^* \in L^1(I)$ such that $\|x^*\|_1 \leq \|f\|_1 + 1/2$, where $\|\cdot\|_1$ is the usual norm of $L^1(I)$.

We claim that if there is $0 < R_0 < 1/2$ such that

$$\frac{\max\{q(t) : t \in I\}}{R_0} + \ln(2) \leq 1, \tag{4.6}$$

then the equation (4.5) has some solution in $\Omega := \{x \in X : \|x\|_\infty \leq R_0\}$.

Indeed, with the notation used above, we have

$$f(t, x(t)) = x(t), \quad p_1 = p_2 \equiv 1, \quad K_1 \equiv 0, \quad K_2(t, s, x(s)) = \frac{t}{t+s} x(s).$$

Therefore, in view of (4.6), conditions (C1) and (C2) hold for $\psi(R) = R$, $\psi_2(R) = R \ln 2$, $\psi_1(R) = 0$. Taking $\beta := 1/\ln(2)$ and $h(r) := r$,

$$\phi_d(\{f(\cdot, x(\cdot)) : x \in S\}) = \phi_d(S) < \beta\phi_d(S),$$

and then the inequality

$$\beta \sup\left\{\left| \int_0^{p_2(t)} K_2(t, s, x(s))ds \right| : x \in S\right\}\phi_d(S) \leq \beta R_0 \phi_d(S) \ln 2 < \frac{\phi_d(S)}{2},$$

holds. Therefore the claim follows by Proposition 4.1.

Acknowledgments

The authors are grateful to the anonymous referees for their useful comments and suggestions, which have improved the quality of the paper.

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